

On the wrapping correction to single magnon energy in twisted $\mathcal{N} = 4$ SYM

Matteo Beccaria and Gian Fabrizio De Angelis

Physics Department, Salento University, Via Arnesano, 73100 Lecce
INFN, Sezione di Lecce
E-mail: matteo.beccaria@le.infn.it

ABSTRACT: We apply Zeilberger summation to derive a closed formula for the wrapping correction to one-impurity states in the $\mathfrak{su}(2)$ sector of the β -deformed $\mathcal{N} = 4$ SYM theory at $\beta = 1/2$. As an application depending heavily on the result, we compute the large volume expansion of the wrapping correction.

Contents

1. Introduction and result	1
2. The wrapping correction from superspace diagrams	3
3. The wrapping correction from unphysical undeformed spectrum	4
4. Proof of the closed formula	5
4.1 Step 1: Implicit formula for the Laurent coefficients	5
4.2 Step 2: Performing the finite summation	6
4.2.1 Algorithms for hypergeometric summation	6
4.2.2 Application to $h_{L,\ell}$	7
4.2.3 Closed form of the Laurent expansion	8
4.3 Step 3: Vanishing of the rational part	9
5. An application of the closed formula: The large L expansion	9
A. Examples of Zeilberger's summation	11
B. List of wrapping corrections	13

1. Introduction and result

A major outcome of AdS/CFT duality [1] is the non perturbative integrability structure of the four dimensional planar $\mathcal{N} = 4$ SYM theory. As a consequence, the calculation of higher-loop anomalous dimensions of single trace gauge invariant composite operators can be performed starting from the asymptotic long-range Bethe Ansatz equations [2]. In the generalized spin chain interpretation [3], the long-range equations reproduce perturbation theory by means of a sequence of integrable Hamiltonians whose interaction range increases with the loop order [4]. Thus, since every operator is associated with a specific chain length, at a certain loop order wrapping effects enter the calculation. Since they are not included in the asymptotic Bethe Ansatz, their computation is a mandatory ingredient in the determination of the spectrum.

In the past years, an intense activity has been devoted to a complete determination of the leading order wrapping effects for various so-called twist operators [5]. This has been feasible by means of generalized Lüscher finite size corrections [6] made suitable for the AdS/CFT case. Besides, very recently, more complete treatments have been proposed which in principle can go beyond the leading order and cover the full $\mathfrak{psu}(2, 2|4)$ invariant theory [7].

Moving away from the maximally symmetric $\mathcal{N} = 4$ SYM theory, it would be interesting to extend wrapping calculations to deformed models with less symmetry, but unbroken all-loop integrability. An important theoretical laboratory of this kind is the superconformal $\mathcal{N} = 1$ β -deformation of $\mathcal{N} = 4$ SYM at real β [9]. This deformed theory is dual to string propagation on the Lunin-Maldacena background [10]. Integrability at all-orders is known to be preserved by the deformation [11, 12].

The simplest state that can be analyzed in this theory is the length- L one-impurity operator in the $\mathfrak{su}(2)_\beta$ sector. It takes the simple form

$$\mathcal{O}_L = \text{Tr}(\varphi Z^{L-1}), \quad (1.1)$$

where φ and Z are the two elementary scalars appearing in the $\mathfrak{su}(2)_\beta$ composite operators. For $L > 2$ the anomalous dimension of this operator is not protected by supersymmetry and reads [9]

$$\gamma_L^\beta = \gamma_{L,\text{asympt}}^\beta + \gamma_{L,\text{wrap}}^\beta, \quad (1.2)$$

$$\gamma_{L,\text{asympt}}^\beta = -1 + \sqrt{1 + 16 g^2 \sin^2(\pi \beta)}, \quad (1.3)$$

$$\gamma_{L,\text{wrap}}^\beta = g^{2L} \delta\gamma_L^\beta + \dots, \quad (1.4)$$

where $g^2 = \frac{g_{\text{YM}}^2 N_c}{16\pi^2}$ is the planar coupling. The piece $g^{2L} \delta\gamma_L^\beta$ is the leading order wrapping correction which has been discussed in great details in [15, 16]. By direct diagram calculations in superspace, it is possible to compute $\delta\gamma_L^\beta$ for each given L by solving a recursion relation between multi-loop basic diagrams. This construction is carried over up to $L = 9$ revealing a very clean transcendentality structure of the result.

A major simplification occurs when $\beta = 1/2$ and L is even. This is in a sense the most simple twisting of $\mathcal{N} = 4$. Indeed, the Bethe Ansatz equations are the same as in the undeformed case $\beta = 0$ apart from a change in the cyclicity constraint for single trace operators. As a consequence, following the analysis of [14, 13], the wrapping correction $\delta\gamma_L^{1/2}$ can be computed in the undeformed theory with relaxed cyclicity constraint in terms of the wrapping correction to a single magnon in $\mathfrak{sl}(2)$ sector with unphysical momentum $p = \pi$. This is a calculation that can be done with the techniques of [5]. The result is an efficient algebraic algorithm for the calculation of $\delta\gamma_L^{1/2}$ which provides a quick answer for each L . This method agrees in all available cases, with the superspace calculation and has the advantage of bypassing the recursive evaluation of multi-loop diagrams.

Nevertheless, apart from the maximal transcendentality term in the answer, we still do not get a closed formula for the correction. The aim of this brief paper is that of providing such formula. It is very simple and compact and reads (L even)

$$\delta\gamma_L^{1/2} = -64 \binom{2L-3}{L-1} \zeta(2L-3) + 128 \sum_{\ell=1}^{\frac{L}{2}-1} \frac{2^{2\ell} \ell}{L-2\ell-1} \binom{2L-2\ell-3}{L-1} \zeta(2L-2\ell-3), \quad (1.5)$$

Specific cases are reported in Appendix B. As a simple application of Eq. (1.5), we shall compute the large L behavior of the correction. This is a quantity which crucially needs

the closed result Eq. (1.5). The leading term in the large volume expansion turns out to be $\delta\gamma_L^{1/2} \sim 4^L L^{-3/2}$ giving an exponentially vanishing *critical* wrapping correction if $g^2 < 1/4$. Notice that the exponential term is shared by the maximal transcendentality contribution, but the algebraic correcting factor is different due to various cancellations in Eq. (1.5).

In our opinion, the derivation of Eq. (1.5) is not trivial. In particular, we needed some sophisticated summation algorithms that, we believe, could deserve a certain interest by themselves. For this reason, we have tried to present the necessary tools in a self-contained way with some illustrative simple example.

2. The wrapping correction from superspace diagrams

As we recalled in the Introduction, the main result of [16] is the first wrapping correction $\delta\gamma_L^\beta g^{2L}$ to the anomalous dimension of the 1-impurity operator $\text{Tr}(\varphi Z^{L-1})$ in the $\mathfrak{su}(2)$ sector of the β -deformed $N = 4$ SYM theory. Here, we shall be interested in the $\beta = 1/2$ case where the undeformed theory is twisted in the simplest way. We shall also restrict our analysis to the case of even L for technical reasons which will be clearer later. For $\beta = 1/2$ and even L the wrapping correction reads

$$\delta\gamma_L^{1/2} = -32 L \left(P_L - 2 \sum_{j=0}^{\lfloor \frac{L}{2} \rfloor - 1} (-1)^j I_L^{(j+1)} \right), \quad (2.1)$$

$$P_L = \frac{2}{L} \binom{2L-3}{L-1} \zeta(2L-3), \quad (2.2)$$

where I_L^j can be recursively extracted from multi-loop diagrams and are explicitly tabulated up to $L = 9$ in [16]. For the present discussion it is important to remark that no general closed expression is available for these crucial quantities. In other words, the quantity $\delta\gamma_L^{1/2}$ can be in principle be computed for any given L , but not as a closed function of this physically meaningful parameter.

From inspection of the first cases, $L = 4, 6, 8, \dots$, one is led to *conjecture* the following general form of the wrapping correction

$$\delta\gamma_L^{1/2} = \sum_{\ell=0}^{\frac{L}{2}-1} a_{L,\ell} \zeta(2L-3-2\ell), \quad (2.3)$$

with *integer* coefficients $a_{L,\ell}$. Also, the maximum transcendentality term comes entirely from the P_L term in Eq. (2.1). This means that one coefficient is known

$$a_{L,0} = -64 \binom{2L-3}{L-1}. \quad (2.4)$$

3. The wrapping correction from unphysical undeformed spectrum

In the approach pursued in [13, 14], the wrapping correction of Eq. (2.1) can be identified with the wrapping correction to the energy of a single unphysical magnon with momentum $p = \pi$ in the undeformed theory. This can be computed by Lüscher-Janik finite size corrections in the $\mathfrak{sl}(2)$ sector.

The resulting formula is quite simple ¹

$$\delta\gamma_L^{1/2} = -8^2 \cdot 4^{L-2} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{T^2(q, Q)}{R(q, Q)} \frac{1}{(q^2 + Q^2)^{L-2}}, \quad (3.1)$$

where $T(q, Q)$ and $R(q, Q)$ are polynomials in q and Q which can be expressed in terms of the (almost trivial) Baxter function associated with the single magnon state with $p = \pi$. Their general expression can be found in [13] and will not be needed here. Kinematical arguments show that, under summation over Q , the integral over the rapidity q is given by the residue at the pole $q = iQ$. Thus we can write

$$\begin{aligned} \delta\gamma_L^{1/2} &= -4^{L+1} i \sum_{Q=1}^{\infty} \text{Res}_{q=iQ} \left(\frac{T^2(q, Q)}{R(q, Q)} \frac{1}{(q^2 + Q^2)^{L-2}} \right) = \\ &= -4^{L+1} i \sum_{Q=1}^{\infty} \text{Res}_{q=iQ} \Lambda_L(q, Q), \end{aligned} \quad (3.2)$$

where $\Lambda_L(q, Q)$ is

$$\Lambda_L(q, Q) = \frac{T^2(q, Q)}{R(q, Q)} \frac{1}{(q^2 + Q^2)^{L-2}}. \quad (3.3)$$

Replacing the explicit expressions of T , and R we have

$$\Lambda_L(q, Q) = \frac{4Q^2(q^2 + Q^2 - 1)^2}{(q^2 + Q^2)^L [q^2 + (Q+1)^2] [q^2 + (Q-1)^2]}. \quad (3.4)$$

On general grounds we can easily prove that

$$\text{Res}_{q=iQ} \Lambda_L(q, Q) = \frac{P_L(Q)}{Q^{2L-3}(4Q^2 - 1)^L}, \quad (3.5)$$

where $P_L(Q)$ is an even polynomial of degree $\deg P_L = 3L - 2$. Again, one can inspect the first cases, $L = 4, 6, 8, \dots$. In all instances, the residue can be written as

$$\frac{P_L(Q)}{Q^{2L-3}(4Q^2 - 1)^L} = \sum_{\ell=0}^{\frac{L}{2}-1} \frac{a'_{L,\ell}}{Q^{2L-3-2\ell}} + R_L(Q), \quad (3.6)$$

where $R_L(Q)$ is a rational function regular in $Q = 0$ and such that

$$\sum_{Q=1}^{\infty} R_L(Q) = 0. \quad (3.7)$$

The explicit value of the coefficients of the polar part obeys $a_{L,\ell} = a'_{L,\ell}$ and matches perfectly those in Eq. (2.3)

¹The factor 8^2 is the square of the one magnon one-loop anomalous dimension

4. Proof of the closed formula

Our aim, will be that of establishing Eq. (2.3) rigorously with a simple closed formula for $a_{L,\ell}$. To this aim, we shall pursue the residue formula described in the previous section and perform the following 3 steps:

1. We compute the coefficients of the Laurent expansion of $\text{Res}_{q=iQ} \Lambda_L(q, Q)$ in $Q = 0$ as complicated finite sums involving $\mathcal{O}(L)$ terms.
2. We find a closed form of the poles coefficients by application of the Zeilberger's summation algorithm. For the polar part, these coefficients are clearly in one to one relation with the coefficients $a_{L,\ell}$ in Eq. (2.3).
3. We prove that the rational function R_L appearing in Eq. (3.6) obeys indeed Eq. (3.7).

4.1 Step 1: Implicit formula for the Laurent coefficients

We start from the trivial decomposition

$$\Lambda_L(q, Q) = \frac{4Q^2}{(q^2 + Q^2)^L} + \tilde{\Lambda}_L(q, Q) + \tilde{\Lambda}_L(q, -Q), \quad (4.1)$$

where

$$\tilde{\Lambda}_L(q, Q) = -\frac{4Q(Q+1)^2}{q^2 + (Q+1)^2} \frac{1}{(q^2 + Q^2)^L}. \quad (4.2)$$

The first term in Eq. (4.1) is responsible for the maximal transcendentality contribution as discussed in [13]. Its contribution is known, see Eq. (2.4), and we focus on the other terms. We set $q = iQ + zQ$ and take the Laurent expansion of Eq. (4.2) around $z = 0$

$$\tilde{\Lambda}(iQ + zQ, Q) = \frac{-4(Q+1)^2}{Q^{2L-1} \omega^L (1 + 2Q + Q^2 \omega)}, \quad (4.3)$$

where

$$\omega = z^2 + 2iz = 2iz \left(1 - i\frac{z}{2}\right). \quad (4.4)$$

Expanding first in powers of ω and then in powers of z we find

$$\tilde{\Lambda}(iQ + zQ, Q) = -\frac{4(Q+1)^2}{Q^{2L-1}} \sum_{k \geq 0} (-1)^k \frac{Q^{2k}}{(1+2Q)^{k+1}} (2iz)^{k-L} \sum_{p \geq 0} \binom{k-L}{p} \left(\frac{z}{2i}\right)^p. \quad (4.5)$$

We can now extract the coefficient \tilde{c}_L of the simple pole $\tilde{\Lambda}(iQ + zQ, Q) = \dots + \tilde{c}_L/z + \dots$. It is Q times the residue in the initial q variable. It reads

$$\tilde{c}_L = -\frac{8i(Q+1)^2}{Q^{2L-1}} 2^{-2L} \sum_{k \geq 0} 2^{2k} \frac{Q^{2k}}{(1+2Q)^{k+1}} \binom{k-L}{L-k-1}. \quad (4.6)$$

Expanding the powers of $1/(1+2Q)$, we find

$$\tilde{c}_L = -\frac{i2^{3-2L}(Q+1)^2}{Q^{2L-1}} \sum_{k \geq 0} \sum_{p \geq 0} (2Q)^{2k+p} \binom{k-L}{L-k-1} \binom{-k-1}{p}. \quad (4.7)$$

The sum can be reorganized as

$$\tilde{c}_L = -i 2^{4-2L} \sum_{\ell=1}^{\infty} \frac{2^{2\ell}}{Q^{2L-2\ell-2}} \sum_{k=0}^{\ell} \binom{k-L}{L-k-1} \times \quad (4.8)$$

$$\times \left[\binom{-k-1}{2\ell-2k+1} + \binom{-k-1}{2\ell-2k} + \frac{1}{4} \binom{-k-1}{2\ell-2k-1} \right] = \quad (4.9)$$

$$= -i 2^{4-2L} \sum_{\ell=1}^{\infty} \frac{2^{2\ell}}{Q^{2L-2\ell-2}} h_{L,\ell}, \quad (4.10)$$

where

$$h_{L,\ell} = \sum_{k=0}^{\ell} \binom{k-L}{L-k-1} \binom{-k-1}{2\ell-2k} \frac{k-\ell-2\ell^2}{2(2\ell-k)(2\ell-2k+1)}. \quad (4.11)$$

This is a rather complicated finite sum. In practice, to obtain a closed formula for the wrapping correction is completely equivalent to finding the same for this sum. Indeed, in this section, we have just shown that

$$\delta\gamma_L^{1/2} = \sum_{Q=1}^{\infty} \delta\gamma_L^{1/2}(Q), \quad (4.12)$$

with

$$\delta\gamma_L^{1/2}(Q) = -64 \binom{2L-3}{L-1} \frac{1}{Q^{2L-3}} - 128 \sum_{\ell=1}^{\infty} \frac{2^{2\ell}}{Q^{2L-2\ell-3}} h_{L,\ell}. \quad (4.13)$$

4.2 Step 2: Performing the finite summation

Up to now, we have just reshuffled the Lüscher-Janik's correction formula. The novelty comes when one tries to perform in closed form the finite sum defining $h_{L,\ell}$, namely Eq. (4.11). This can be done by applying a powerful tool, the Zeilberger's algorithm [17]. It is a very nice mathematical device to perform rather difficult finite summations. We now briefly describe it referring to [18] for a more detailed discussion. Then, we present its application to Eq. (4.11).

4.2.1 Algorithms for hypergeometric summation

Let us consider the finite sum

$$S_L = \sum_{k \in \mathbb{Z}} \sigma_{L,k}, \quad (4.14)$$

where $\sigma_{L,k} \neq 0$ in a L -dependent finite interval $k_{\min}(L) \leq k \leq k_{\max}(L)$. Is it possible to write the sum in closed form as a function of L ? A general strategy to face this problem is available when $\sigma_{L,k}$ is doubly hypergeometric, *i.e.* the ratios $\sigma_{L+1,k}/\sigma_{L,k}$ and $\sigma_{L,k+1}/\sigma_{L,k}$ are rational functions of L and k . The strategy is known as Sister Celine's algorithm and consists in finding positive integers I, J such that

$$\sum_{i=0}^I \sum_{j=0}^J c_{i,j}(L) \sigma_{L+i,k+j} = 0. \quad (4.15)$$

Basically, one divides by $\sigma_{L,k}$, simplifies, put everything under a common denominator, and equates to zero the various powers of k in the numerator. Under very mild conditions, such a doubly recursive relation can always be found if I and J are taken large enough. Summing over k , we derive a recursion for the finite sum

$$\sum_{i=0}^I \left(\sum_{j=0}^J c_{i,j}(L) \right) S_{L+i} = 0. \quad (4.16)$$

This can be used to find a closed form for Eq. (4.14) or, possibly, to show that a closed form does not exist in a given class²

Zeilberger's summation algorithm is a very efficient version of Sister Celine's one. It provides an instance of Eq. (4.15) in the nice form

$$\mathbb{D}_L \sigma_{L,k} = \Delta_k(\sigma_{L,k} R_{L,k}), \quad (4.17)$$

where

$$\mathbb{D}_L f_L = \sum_{i=0}^I c_i(L) f_{L+i}, \quad (4.18)$$

$$\Delta_k f_k = f_{k+1} - f_k, \quad (4.19)$$

and $R_{L,k}$ is a rational function. By summing over k , this relation immediately provides the desired recursion relation

$$\mathbb{D}_L S_L = 0, \quad (4.20)$$

since the right hand side telescopes. Explicit illustrative examples are collected in Appendix A.

4.2.2 Application to $h_{L,\ell}$

In order to apply Zeilberger's algorithm it is necessary to modify a little the presentation of $h_{L,k}$. To this aim, we use the following identity valid for the usual extension of the binomial coefficient

$$\binom{a}{b} = (-1)^b \binom{b-a-1}{b}, \quad (4.21)$$

and write

$$h_{L,\ell} = \sum_{k=0}^{\ell} \sigma_{L,\ell,k}, \quad (4.22)$$

with

$$\sigma_{L,\ell,k} = -(-1)^k \binom{2L-2k-2}{L-k-1} \binom{2\ell-k}{2\ell-2k} \frac{k-\ell-2\ell^2}{2(2\ell-k)(2\ell+1-2k)}. \quad (4.23)$$

²An example is the class of finite linear combination of hypergeometric terms, fully treated in [19].

A double application of Zeilberger's algorithm with respect to L or ℓ provides the following two recursions

$$(1 + \ell)(1 + 2\ell - L)(2 + 2\ell - L)\sigma_{L,\ell,k} \quad (4.24)$$

$$\begin{aligned} -2\ell(3 + 2\ell - 2L)(2 + \ell - L)\sigma_{L,\ell+1,k} &= \Delta_k(\sigma_{L,\ell,k} F_{L,\ell,k}), \\ -2(2\ell - 2L + 1)(\ell - L + 1)\sigma_{L,\ell,k} - (2\ell - L)L\sigma_{L+1,\ell,j} &= \Delta_k(\sigma_{L,\ell,k} G_{L,\ell,k}), \end{aligned} \quad (4.25)$$

with

$$\begin{aligned} F_{L,\ell,k} &= \frac{k(2\ell - k)(-2k + 2L - 1)}{2(-k + \ell + 1)(-2k + 2\ell + 3)(2\ell^2 + \ell - k)} \left[2(3\ell - 2L + 4)k^2 + \right. \\ &\quad + 2(6\ell^3 + (3 - 4L)\ell^2 + (3L - 14)\ell + 5L - 10)k + \\ &\quad \left. - 2(8\ell^4 + (18 - 6L)\ell^3 + (3 - 5L)\ell^2 + (4L - 13)\ell + 3L - 6) \right], \end{aligned} \quad (4.26)$$

$$G_{L,\ell,k} = \frac{k(2\ell - k)(2k + 2(2\ell^2 + \ell - 2L))(-2k + 2L - 1)}{(2\ell^2 + \ell - k)(L - k)}. \quad (4.27)$$

As we remarked, Eq. (4.24) and Eq. (4.25) can be checked by direct substitution of Eq. (4.23). Summing over k we find the two recursions for the sum $h_{L,\ell}$

$$\begin{aligned} (1 + \ell)(1 + 2\ell - L)(2 + 2\ell - L)h_{L,\ell} + \\ -2\ell(3 + 2\ell - 2L)(2 + \ell - L)h_{L,\ell+1} &= 0, \end{aligned} \quad (4.28)$$

$$-2(2\ell - 2L + 1)(\ell - L + 1)h_{L,\ell} - (2\ell - L)Lh_{L+1,\ell} = 0. \quad (4.29)$$

In a subset Ω of the discrete (L, ℓ) plane which is maximally connected by the recursion relations, the only solution to Eq. (4.28) and Eq. (4.29) is

$$h_{L,\ell} = C_\Omega \frac{\ell}{L - 2\ell - 1} \binom{2L - 2\ell - 3}{L - 1}, \quad (4.30)$$

where C_Ω is a constant. To prove this statement it is enough to define

$$\widehat{h}_{L,\ell} = h_{L,\ell} \left[\frac{\ell}{L - 2\ell - 1} \binom{2L - 2\ell - 3}{L - 1} \right]^{-1}, \quad (4.31)$$

and check that the two recursions simplify to

$$\widehat{h}_{L,\ell} - \widehat{h}_{L,\ell+1} = 0, \quad (4.32)$$

$$\widehat{h}_{L,\ell} - \widehat{h}_{L+1,\ell} = 0, \quad (4.33)$$

showing that $\widehat{h}_{L,\ell}$ is constant over any connected region.

4.2.3 Closed form of the Laurent expansion

We are interested in the cases $\ell \geq 1$. It follows that there are two disconnected regions Ω . The first is $1 \leq \ell \leq \frac{L}{2} - 1$ and is associated with the polar part of the Laurent expansion.

The second is $\ell \geq L - 1$ and defines an infinite power series, regular around $Q = 0$. Fixing C_Ω by means of one special value in each region, we find the final result for $\delta\gamma_L^{1/2}(Q)$

$$\begin{aligned}\delta\gamma_L^{1/2}(Q) &= \text{pol}_L(Q) + \text{reg}_L(Q), \\ \text{pol}_L(Q) &= -64 \binom{2L-3}{L-1} \frac{1}{Q^{2L-3}} + 128 \sum_{\ell=1}^{\frac{L}{2}-1} \frac{2^{2\ell}\ell}{L-2\ell-1} \binom{2L-2\ell-3}{L-1} \frac{1}{Q^{2L-2\ell-3}}, \\ \text{reg}_L(Q) &= 16 \cdot 2^{2L} \sum_{\ell=0}^{\infty} 2^{2\ell} \frac{L-1+\ell}{L-1+2\ell} \binom{L+2\ell-1}{L-1} Q^{2\ell+1}.\end{aligned}\tag{4.34}$$

After sum over Q , the polar part gives precisely the result Eq. (1.5). To finish, we have just to show that the regular part does not give any contribution. This is shown in the next section.

4.3 Step 3: Vanishing of the rational part

We want to prove that

$$\sum_{Q=1}^{\infty} \text{reg}_L(Q) = 0.\tag{4.35}$$

This follows from the remarkable identity

$$\text{reg}_L(Q) = 2^{2L+3} Q \left[\frac{Q+1}{(2Q+1)^L} - \frac{Q-1}{(2Q-1)^L} \right].\tag{4.36}$$

Eq. (4.36) is easily proved since (a) the Taylor expansion around $Q = 0$ of the right hand side coincides with the regular expansion in Eq. (4.34), and (b) we know a priori that $\text{reg}_L(Q)$ is a rational function of Q . Eq. (4.36) follows from (a)+(b) and the fact that a rational function is completely determined by its Laurent expansion around any point. Taking the sum over Q , we can write

$$\sum_{Q=1}^{\infty} \text{reg}_L(Q) = \sum_{Q=1}^{\infty} [F(Q+1) - F(Q)], \quad F(Q) = 2^{2L+3} \frac{Q(Q-1)}{(2Q-1)^L},\tag{4.37}$$

which clearly vanishes for the physically relevant cases $L > 2$.

5. An application of the closed formula: The large L expansion

As an application of Eq. (1.5), let us derive the large L asymptotics of the critical wrapping correction. At large L we can definitely replace the ζ functions by 1 up to exponentially small terms. We thus have to compute the large L expansion of the two pieces

$$A_L = -64 \binom{2L-3}{L-1},\tag{5.1}$$

$$B_L = 128 \sum_{\ell=1}^{\frac{L}{2}-1} \frac{2^{2\ell}\ell}{L-2\ell-1} \binom{2L-2\ell-3}{L-1}.\tag{5.2}$$

The expansion of the first term is trivial, by Stirling large L expansion

$$A_L = \frac{4^L}{\sqrt{\pi}} \left(-\frac{8}{L^{1/2}} - \frac{3}{L^{3/2}} - \frac{25}{16} \frac{1}{L^{5/2}} + \dots \right). \quad (5.3)$$

The second term is more involved. Another application of Zeilberger's algorithm puts it in the nicer form

$$B_L = 8 \left(\frac{16}{9} \right)^{L/2-1} \sum_{k=0}^{\frac{L}{2}-2} \left(\frac{9}{16} \right)^k \frac{8k+9}{k+1} \binom{4k+4}{2k+1}. \quad (5.4)$$

Expanding the summand, and dropping the term $k = 0$ which is exponentially suppressed with respect to the full sum, we have

$$B_L = \frac{8}{\sqrt{2}\pi} \left(\frac{16}{9} \right)^{L/2-1} \sum_{k=1}^{\frac{L}{2}-2} 9^k \left(\frac{128}{k^{1/2}} - \frac{120}{k^{3/2}} + \frac{637}{4} \frac{1}{k^{5/2}} + \dots \right). \quad (5.5)$$

The sums are

$$\sum_{k=1}^{\frac{L}{2}-2} \frac{9^k}{k^s} = -9^{\frac{L}{2}-1} \Phi \left(9, s, \frac{L}{2} - 1 \right) + \text{Li}_s(9). \quad (5.6)$$

where Φ is the Lerch transcendent function

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}. \quad (5.7)$$

The polylogarithm contribution is exponentially suppressed. Dropping it, we find

$$B_L = \frac{1}{2\sqrt{2}\pi} 4^L \left(-128 \Phi \left(9, \frac{1}{2}, \frac{L}{2} - 1 \right) + 120 \Phi \left(9, \frac{3}{2}, \frac{L}{2} - 1 \right) - \frac{637}{4} \Phi \left(9, \frac{5}{2}, \frac{L}{2} - 1 \right) + \dots \right). \quad (5.8)$$

The large L expansion of these Lerch functions can be obtained from the large N expansion of its integral representation valid for $z < 1$,

$$\Phi(z, s, N) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-Nt}}{1 - z e^{-t}} dt, \quad (5.9)$$

and analytically continuing to $z = 9$. This gives

$$\Phi \left(9, \frac{1}{2}, N \right) = -\frac{1}{8} \frac{1}{N^{1/2}} - \frac{9}{128} \frac{1}{N^{3/2}} - \frac{135}{2048} \frac{1}{N^{5/2}} + \dots, \quad (5.10)$$

$$\Phi \left(9, \frac{3}{2}, N \right) = -\frac{1}{8} \frac{1}{N^{3/2}} - \frac{27}{128} \frac{1}{N^{5/2}} + \dots, \quad (5.11)$$

$$\Phi \left(9, \frac{5}{2}, N \right) = -\frac{1}{8} \frac{1}{N^{5/2}} + \dots. \quad (5.12)$$

and therefore,

$$B_L = \frac{4^L}{\sqrt{\pi}} \left(+\frac{8}{L^{1/2}} + \frac{2}{L^{3/2}} + \frac{1}{16} \frac{1}{L^{5/2}} + \dots \right). \quad (5.13)$$

Summing the A_L contribution we see that the leading term cancels and we end up with

$$\delta\gamma_L^{1/2} = -\frac{4^L}{L^{3/2}\sqrt{\pi}} \left(1 + \frac{3}{2L} + \dots\right). \quad (5.14)$$

This expansion is perfectly matched by the numerical values that we could compute at very large L thanks to the closed formula. As a remark, we emphasize that an estimate based on the only maximal transcendentality term would have predicted the correct exponential factor 4^L , but a wrong algebraic correction $\sim L^{-1/2}$ instead of the correct one $\sim L^{-3/2}$.

Acknowledgments

We thank N. Gromov and P. Vieira for useful conversations on twisted $\mathcal{N} = 4$ SYM.

A. Examples of Zeilberger's summation

As illustration we can consider the following interesting examples

$$S_L^{(p)} = \sum_{k=0}^L (-1)^k \binom{L}{k}^p, \quad p = 1, 2, 3, \quad (A.1)$$

which can be summed as follows

$$S_L^{(1)} = 0, \quad (A.2)$$

$$S_{2L}^{(2)} = (-1)^L 2^L \frac{(2L-1)!!}{L!}, \quad S_{2L+1}^{(2)} = 0, \quad (A.3)$$

$$S_{2L}^{(3)} = (-1)^L \frac{(3L)!}{(L!)^3}, \quad S_{2L+1}^{(3)} = 0. \quad (A.4)$$

The first is trivial, the second easy, the third quite difficult. Let us treat them symmetrically with the Zeilberger's algorithm. In all cases, we extend the sum and write

$$S_L^{(p)} = \sum_{k \in \mathbb{Z}} \sigma_{L,k}^{(p)}, \quad \sigma_{L,k}^{(p)} = (-1)^k \binom{L}{k}^p. \quad (A.5)$$

$p = 1$

For $\sigma_{L,k}^{(1)}$ Zeilberger's algorithm provides the recursion

$$\sigma_{L,k}^{(1)} = \Delta_k \left(-\frac{k}{L} \sigma_{L,k}^{(1)} \right). \quad (A.6)$$

Explicitly, this is just Pascal's relation

$$\binom{L}{k} = \frac{k+1}{L} \binom{L}{k+1} + \frac{k}{L} \binom{L}{k} = \binom{L-1}{k} + \binom{L-1}{k-1}. \quad (A.7)$$

Multiplying by $(-1)^k$ and summing over k , we immediately obtain $S_L^{(1)} = 0$.

$p = 2$

For $\sigma_{L,k}^{(2)}$ Zeilberger's algorithm provides the recursion

$$(L+2)\sigma_{L+2,k}^{(2)} + 4(L+1)\sigma_{L,k}^{(2)} = \Delta_k(\sigma_{L,k}^{(2)} R_{L,k}), \quad (\text{A.8})$$

with

$$R_{L,k} = -\frac{k^2}{(L-k+1)^2(L-k+2)^2} \left[2(L+1)k^2 - 2(L+1)(3L+5)k + (L+1)(5L^2 + 17L + 14) \right]. \quad (\text{A.9})$$

Summing over k we find the non-trivial recursion relation

$$(L+2)S_{L+2}^{(2)} + 4(L+1)S_L^{(2)} = 0. \quad (\text{A.10})$$

Using the initial values, we obtain Eq. (A.3).

$p = 3$

Finally, in the case $p = 3$, Zeilberger's algorithm gives

$$(L+2)^2\sigma_{L+2,k}^{(3)} + 3(3L+4)(3L+2)\sigma_{L,k}^{(3)} = \Delta_k(\sigma_{L,k}^{(3)} R_{L,k}), \quad (\text{A.11})$$

with

$$R_{L,k} = -\frac{k^3}{(-k+L+1)^3(-k+L+2)^3} \left[3(3L+4)k^4 - 3(3L+4)(5L+8)k^3 + 3(29L^3 + 132L^2 + 198L + 98)k^2 - 3(L+2)(26L^3 + 109L^2 + 151L + 69)k + 2(L+2)^2(14L^3 + 54L^2 + 69L + 29) \right]. \quad (\text{A.12})$$

Summing over k , we find the recursion

$$(L+2)^2 S_{L+2} + 3(3L+4)(3L+2) S_L = 0, \quad (\text{A.13})$$

from which the result Eq. (A.4) follows.

B. List of wrapping corrections

$$\begin{aligned}
\delta\gamma_4^{1/2} &= 128 (4\zeta_3 - 5\zeta_5), \\
\delta\gamma_6^{1/2} &= 128 (32\zeta_5 + 28\zeta_7 - 63\zeta_9), \\
\delta\gamma_8^{1/2} &= 768 (32\zeta_7 + 64\zeta_9 + 44\zeta_{11} - 143\zeta_{13}), \\
\delta\gamma_{10}^{1/2} &= 128 (1024\zeta_9 + 3520\zeta_{11} + 4576\zeta_{13} + 2860\zeta_{15} - 12155\zeta_{17}), \\
\delta\gamma_{12}^{1/2} &= 256 (2560\zeta_{11} + 13312\zeta_{13} + 26208\zeta_{15} + 28288\zeta_{17} + 16796\zeta_{19} - 88179\zeta_{21}), \\
\delta\gamma_{14}^{1/2} &= 256 (12288\zeta_{13} + 89600\zeta_{15} + 243712\zeta_{17} + 372096\zeta_{19} + 361760\zeta_{21} \\
&\quad + 208012\zeta_{23} - 1300075\zeta_{25}), \\
\delta\gamma_{16}^{1/2} &= 512 (28672\zeta_{15} + 278528\zeta_{17} + 992256\zeta_{19} + 1984512\zeta_{21} + 2615008\zeta_{23} \\
&\quad + 2377280\zeta_{25} + 1337220\zeta_{27} - 9694845\zeta_{29}), \\
\delta\gamma_{18}^{1/2} &= 128 (524288\zeta_{17} + 6537216\zeta_{19} + 29417472\zeta_{21} + 73835520\zeta_{23} + 123059200\zeta_{25} \\
&\quad + 147251520\zeta_{27} + 127743840\zeta_{29} + 70715340\zeta_{31} - 583401555\zeta_{33}), \\
\delta\gamma_{20}^{1/2} &= 256 (1179648\zeta_{19} + 18350080\zeta_{21} + 101556224\zeta_{23} + 310886400\zeta_{25} + 631488000\zeta_{27} \\
&\quad + 932305920\zeta_{29} + 1042120800\zeta_{31} + 873396480\zeta_{33} + 477638700\zeta_{35} - 4418157975\zeta_{37}).
\end{aligned}$$

References

- [1] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998) [*Int. J. Theor. Phys.* **38**, 1113 (1999)] [arXiv:hep-th/9711200].
- [2] J. A. Minahan and K. Zarembo, *The Bethe-ansatz for N = 4 super Yang-Mills*, *JHEP* **0303**, 013 (2003) [arXiv:hep-th/0212208].
N. Beisert, C. Kristjansen and M. Staudacher, *The dilatation operator of N = 4 super Yang-Mills theory*, *Nucl. Phys. B* **664**, 131 (2003) [arXiv:hep-th/0303060].
N. Beisert, *The su(2|3) dynamic spin chain*, *Nucl. Phys. B* **682**, 487 (2004) [arXiv:hep-th/0310252].
M. Staudacher, *The factorized S-matrix of CFT/AdS*, *JHEP* **0505**, 054 (2005) [arXiv:hep-th/0412188].
N. Beisert and M. Staudacher, *Long-range PSU(2, 2|4) Bethe ansatz for gauge theory and strings*, *Nucl. Phys. B* **727**, 1 (2005) [arXiv:hep-th/0504190].
N. Beisert, R. Hernandez and E. Lopez, *A crossing-symmetric phase for AdS₅ × S⁵ strings*, *JHEP* **0611**, 070 (2006) [arXiv:hep-th/0609044].
N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* **0701**, P021 (2007) [arXiv:hep-th/0610251].
- [3] N. Beisert, *The dilatation operator of N = 4 super Yang-Mills theory and integrability*, *Phys. Rept.* **405**, 1 (2005) [arXiv:hep-th/0407277].
- [4] N. Beisert, V. Dippel and M. Staudacher, *A novel long range spin chain and planar N = 4 super Yang-Mills*, *JHEP* **0407**, 075 (2004) [arXiv:hep-th/0405001].

- [5] J. Ambjorn, R. A. Janik and C. Kristjansen, *Wrapping interactions and a new source of corrections to the spin-chain/string duality*, Nucl. Phys. B **736**, 288 (2006) [arXiv:hep-th/0510171].
Z. Bajnok and R. A. Janik, *Four-loop perturbative Konishi from strings and finite size effects for multiparticle states*, Nucl. Phys. B **807**, 625 (2009) [arXiv:0807.0399].
Y. Hatsuda and R. Suzuki, *Finite-Size Effects for Multi-Magnon States*, JHEP **0809**, 025 (2008) [arXiv:0807.0643].
Z. Bajnok, R. A. Janik and T. Lukowski, *Four loop twist two, BFKL, wrapping and strings*, arXiv:0811.4448.
A. V. Belitsky, *Baxter equation beyond wrapping*, arXiv:0902.3198 [hep-th].
M. Beccaria, V. Forini, T. Lukowski and S. Zieme, *Twist-three at five loops, Bethe Ansatz and wrapping*, arXiv:0901.4864.
- [6] M. Luscher, *Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories. 2. Scattering States*, Commun. Math. Phys. **105**, 153 (1986).
M. Luscher, *Volume Dependence Of The Energy Spectrum In Massive Quantum Field Theories. 1. Stable Particle States*, Commun. Math. Phys. **104**, 177 (1986).
- [7] N. Gromov, V. Kazakov, A. Kozak and P. Vieira, *Integrability for the Full Spectrum of Planar AdS/CFT II*, arXiv:0902.4458 [].
N. Gromov, V. Kazakov and P. Vieira, *Integrability for the Full Spectrum of Planar AdS/CFT*, arXiv:0901.3753 [].
- [8] D. Bombardelli, D. Fioravanti and R. Tateo, *Thermodynamic Bethe Ansatz for planar AdS/CFT: a proposal*, arXiv:0902.3930 [].
G. Arutyunov and S. Frolov, *Thermodynamic Bethe Ansatz for the $AdS_5 \times S^5$ Mirror Model*, arXiv:0903.0141 [hep-th].
- [9] R. G. Leigh and M. J. Strassler, *Exactly Marginal Operators And Duality In Four-Dimensional $N=1$ Supersymmetric Gauge Theory*, Nucl. Phys. B **447**, 95 (1995) [arXiv:hep-th/9503121].
A. Mauri, S. Penati, A. Santambrogio and D. Zanon, *Exact results in planar $N = 1$ superconformal Yang-Mills theory*, JHEP **0511**, 024 (2005) [arXiv:hep-th/0507282].
- [10] O. Lunin and J. M. Maldacena, *Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity duals*, JHEP **0505**, 033 (2005) [arXiv:hep-th/0502086].
- [11] S. A. Frolov, R. Roiban and A. A. Tseytlin, *Gauge - string duality for superconformal deformations of $N = 4$ super Yang-Mills theory*, JHEP **0507**, 045 (2005) [arXiv:hep-th/0503192].
D. Berenstein and S. A. Cherkis, *Deformations of $N = 4$ SYM and integrable spin chain models*, Nucl. Phys. B **702**, 49 (2004) [arXiv:hep-th/0405215].
- [12] N. Beisert and R. Roiban, *Beauty and the twist: The Bethe ansatz for twisted $N = 4$ SYM*, JHEP **0508**, 039 (2005) [arXiv:hep-th/0505187].
- [13] J. Gunneson, *Wrapping in maximally supersymmetric and marginally deformed $N=4$ Yang-Mills*, arXiv:0902.1427 [hep-th].
- [14] C. Gomez, J. Gunneson and R. Hernandez, *Magnons and BFKL*, JHEP **0809**, 060 (2008) [arXiv:0807.2339 [hep-th]].
- [15] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, JHEP **0808**, 057 (2008) [arXiv:0806.2103 [hep-th]].

- [16] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, *Single impurity operators at critical wrapping order in the beta-deformed $N=4$ SYM*, arXiv:0811.4594 [hep-th].
- [17] D. Zeilberger, *A fast algorithm for proving terminating hypergeometric identities*, Discrete Math. **80** (1990), 207-211;
D. Zeilberger, *The method of creative telescoping*, J. Symbolic Computation **11** (1991), 195-204.
A Maxima implementation is available at the web address
<http://www.risc.uni-linz.ac.at/research/combinat/risc/software/Zeilberger/>
- [18] M. Petkovšek, H. S. Wilf, and D. Zeilberger, *A=B*, AK Peters, Ltd. (January 1996), ISBN-10: 1568810636; ISBN-13: 978-1568810638.
- [19] M. Petkovšek, *Hypergeometric solutions of linear recurrences with polynomial coefficients*, J. Symbolic Computation, **14** (1992), 243-264.